Recent progress on univariate and multivariate polynomial and spline quasi-interpolants

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Abstract

Polynomial and spline quasi-interpolants (QIs) are practical and effective approximation operators. Among their remarkable properties, let us cite for example: good shape properties, easy computation and evaluation (no linear system to solve), uniform boundedness independently of the degree (polynomials) or of the partition (splines), good approximation order. We shall emphasize new results on various types of univariate and multivariate polynomial or spline QIs, depending on the nature of coefficient functionals, which can be differential, discrete or integral. We shall also present some applications of QIs to numerical methods.

1 Introduction

A quasi-interpolant of f has the general form

$$Qf = \sum_{\alpha \in A} \mu_\alpha(f) B_\alpha,$$

where $\{B_\alpha, \alpha \in A\}$ is a family of polynomials or B-splines forming a partition of unity, and $\{\mu_\alpha(f), \alpha \in A\}$ is a family of linear functionals which are local in the sense that they only use values of f in some neighbourhood of $\Sigma_\alpha = supp(B_\alpha)$. The main interest of QIs is that they provide excellent approximants of functions without solving any linear system of equations. In the literature, one can find the three following types of QIs:
(i) Differential QIs (abbr. DQIs) : the linear functionals are linear combinations of values of derivatives of f at some point in Σ_α.

(ii) Discrete QIs (abbr. dQIs) : the linear functionals are linear combinations of values of f at some points in the neighbourhood of Σ_α.

(iii) Integral QIs (abbr. iQIs) : the linear functionals are linear combinations of weighted mean values of f in the neighbourhood of Σ_α.

We shall present various types of univariate and multivariate polynomial and spline QIs, mainly dQIs and iQIs, which were recently introduced in the literature. For polynomial QIs, we only present QIs which are close to the orifginal Bernstein or Durrmeyer operators (for other types of QIs, see for example [35][36]).

The prototype of *polynomial dQIs* is the classical Bernstein operator

$$B_n f = \sum_{i=0}^{n} f(\frac{i}{n}) b_i^{(n)}$$

where $\{b_i^{(n)}(x) = C_n^i x^i (1-x)^{n-i}, 0 \leq i \leq n\}$ is the Bernstein basis of the space $\mathbb{P}_n$ of polynomials of degree at most n (the C_n^i are binomial coefficients).

The prototype of *polynomial iQIs* is the Durrmeyer operator [33]

$$M_n f = \sum_{i=0}^{n} \langle f, \tilde{b}_i^{(n)} \rangle b_i^{(n)}$$

where $\tilde{b}_i^{(n)} = b_i^{(n)} / \int_0^1 b_i^{(n)} = (n+1) b_i^{(n)}$ and $\langle f, g \rangle = \int_0^1 fg$. Both can be extended to the multivariate case, either on the hypercube or on the simplex. Another extension consists in adding a Jacobi weight in the scalar product.

The prototypes of *spline DQIs* are de Boor-Fix QIs [11] and their various univariate and multivariate extensions

$$Q f = \sum_{j \in J} \lambda_j(f) B_j.$$

Here $\{B_j \; j \in J\}$ is a family of univariate B-splines of degree m on a nonuniform sequence of knots $\{t_k\}$. Assuming that $\Sigma_j = supp(B_j) = [t_{j-m}, t_{j+1}]$, we set $E_m = \{-m+1, \ldots, 0\}$ and we define $\psi_j(t) = \prod_{r \in E_m}(t_{j+r} - t) \in \mathbb{P}_m$ for all $j \in J$. For any $\tau \in \Sigma_j$, the coefficient functionals are

$$\lambda_j(f) = \frac{1}{(m-1)!} \sum_{l=0}^{m-1} (-1)^{m-l-1} D^{m-l-1} \psi_j(\tau) D^l f(\tau).$$

The prototypes of *spline dQIs* are the various univariate and multivariate extensions of Schoenberg-Marsden operators [52][53].

$$S f = \sum_{j \in J} f(\tau_j) B_j$$

where τ_j is an interior point of $\Sigma_j = supp(B_\alpha)$.

The prototypes of *spline iQIs* are the various univariate and multivariate extensions of operators [21][63]

$$Tf = \sum_{j \in J} \langle f, M_j \rangle B_j,$$

where M_j is a B-spline (which can be different from B_j) normalized by $\int_{\Sigma_j} M_j = 1$.

As emphasized by de Boor ([10], chapter XII), a spline QI defined on non uniform partitions has to be *uniformly bounded independently of the partition* (abbr. UB) in order to be interesting for applications. Therefore, with some coworkers, we have defined various families of QIs satisfying this property and having an infinite norm as small as possible. In general it is difficult to minimize the true norm of the operator, however, it is often possible to minimize an upper bound of this norm: this gives rise to what we have called near-best (abbr. NB) QIs (see [1],[2]-[4],[40]). Numerical applications are still not very much developed. However, QIs can be useful in approximation and estimation [21][22][85], in numerical quadrature [23][73][75], and for the numerical solution of integral or partial differential equations.

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2 Univariate polynomial QIs

2.1 Basic operators

1) The *Bernstein-Stancu QI* [83] is defined for $x \in [0, 1]$ by

$$S_n^{(\alpha)} f(x) = \sum_{i=0}^{n} f(\frac{i}{n}) b_i^{(n)}(x, \alpha)$$

where the *Bernstein-Stancu basis* is defined by $b_k^{(n)}(x, \alpha) = C_n^k \dfrac{(x)_\alpha^k (1-x)_\alpha^{n-k}}{(1)_\alpha^n}$.

Here $(x)_\alpha^k = x(x + \alpha) \ldots (x + (k - 1)\alpha)$, for $\alpha \in \mathbb{R}$. For $\alpha = 0$, we recover the classical Bernstein basis.

2) The *Bernstein- Phillips (or q-Bernstein) QI* ([56]-[59]) is defined for $x \in [0, 1]$ by

$$B_n^q f(x) = \sum_{i=0}^{n} f\left(\frac{[i]}{[n]}\right) b_k^{(n)}(x, q)$$

where the *Bernstein-Phillips or q-Bernstein basis* is defined for $q \neq 1$ by $b_k^{(n)}(x, q) = \Gamma_n^k x^k (1 - x)_q^{n-k}$. Here $[i] = \frac{1-q^i}{1-q}$, $[i]! = \prod_{s=1}^{i}[s]$, $\Gamma_n^k = \frac{[n]!}{[k]![n-k]!}$ and $(x)_q^k = \prod_{s=0}^{k-1}(1 - q^s x)$. For $q = 1$, we recover the classical Bernstein basis.

Using the notation $e_s(x) = x^s$ for monomials, it is easy to prove that all the above QIs $\mathcal{B}_n$ are exact on $\mathbb{P}_1$, i.e. $\mathcal{B}_n e_s = e_s \ \ for \ \ s = 0, 1$. Moreover they are degree

preserving since $\mathcal{B}_n e_s(x) = e_s(x) + r_{s-1}(x, n)$ where $r_{s-1}(x, n)$ is some polynomial of degree at most $s - 1$ depending on n (and eventually on the parameters α, N or q).

QIs————————————————

2.2 Left and right BQIs

All operators $\mathcal{B}_n$ defined above are isomorphisms of $\mathbb{P}_n$. Moreover $\mathcal{B}_n$ and $\mathcal{A}_n = \mathcal{B}_n^{-1}$ can be expressed as linear differential operators with polynomial coefficients

$$\mathcal{B}_n = \sum_{k=0}^{n} \beta_k^{(n)} D^k, \quad \mathcal{A}_n = \sum_{k=0}^{n} \alpha_k^{(n)} D^k,$$

where $D = \frac{d}{dx}$ and the polynomials $\beta_k^{(n)} \in \mathbb{P}_k$ and $\alpha_k^{(n)} \in \mathbb{P}_k$ are defined by simple recursions (see e.g.[64]-[67] for partial results in this sense).

For $0 \leq r \leq n$, we introduce the partial inverses:

$$\mathcal{A}_n^{(r)} = \sum_{k=0}^{r} \alpha_k^{(n)} D^k,$$

and we consider the two families of right and left BQIs:

(RBQI) The right BQIs $\mathcal{B}_n^{[r]} = \mathcal{B}_n \circ \mathcal{A}_n^{(r)}$ are defined for C^r-functions f by

$$\mathcal{B}_n^{[r]} f = \mathcal{B}_n(\mathcal{A}_n^{(r)} f) = \mathcal{B}_n\left(\sum_{k=0}^{r} \alpha_k^{(n)} D^k f\right).$$

(LBQI) The left BQIs $\mathcal{B}_n^{(r)} = \mathcal{A}_n^{(r)} \circ \mathcal{B}_n$ are defined on any (e.g. continuous) function

$$\mathcal{B}_n^{(r)} f = \mathcal{A}_n^{(r)}(\mathcal{B}_n f) = \sum_{k=0}^{r} \alpha_k^{(n)} D^k(\mathcal{B}_n f).$$

By construction, for $0 \leq r \leq n$, the BQIs $\mathcal{B}_n^{[r]}$ and $\mathcal{B}_n^{(r)}$ are exact on the space $\mathbb{P}_r$. Moreover, in many cases, the LBQIs have a uniformly bounded infinite norm, independent on n for each $0 \leq k \leq n$ fixed (see e.g. [70] [86] for some results of this type). From this property are deduced some convergence results (see [30][67]).

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2.3 Kageyama QIs

Kageyama [44] [45] considers Stancu operators for $\alpha \in [-\frac{1}{n}, 0]$

$$S_n^{(-\frac{1}{n})} = \mathcal{L}_n, \quad S_n^{(0)} = B_n,$$

where $\mathcal{L}_n$ is the Lagrange interpolation operator on the uniform partition of $[0,1]$ (this result is due to Mühlbach). Then he truncates at order s the Maclaurin series of $S_n^{(\alpha)}f$ w.r.t. α and he takes the value of this polynomial at $\alpha = -\frac{1}{n}$:

$$\mathcal{K}_n^{(s)}f = \sum_{j=0}^{s} \frac{1}{j!} \frac{(-1)^j}{n^j} \frac{\partial^j}{\partial \alpha^j} \left[S_n^{(\alpha)}f \right]_{\alpha=0}$$

$\mathcal{K}_n^{(0)} = S_n^{(0)} = B_n$ and $\mathcal{K}_n^{(\infty)} = S_n^{(-\frac{1}{n})} = \mathcal{L}_n$. He also gives expansions of $\mathcal{K}_n f$ in terms of derivatives of $B_n f$ and in powers of $\frac{1}{n}$. He proves that, for all s fixed, $\|\mathcal{K}_n^{(s)}\|_\infty$ is uniformly bounded and give Voronovskaja type results, e.g.

$$\lim_{n\to\infty} n^{s+1}(\mathcal{K}_n^{(s)}f - f) = -\sum_{k=0}^{2s+2} \frac{1}{k!} \Upsilon_{s+1,k} D^k f$$

where the polynomials $\Upsilon_{s+1,k}$ can be computed by recursion. He also compares the expansions of $\mathcal{L}_n$, the BQIs $B_n^{(r)}$ and $\mathcal{K}_n^{(s)}$ interms of derivatives of $B_n f$ with polynomial coefficients. Numerical experiments done by the author suggest that these operators are in general better approximants than BQIs of section 3.1.

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2.4 Univariate Durrmeyer and Goodman-Sharma QIs

A straightforward generalization of the Durrmeyer operator M_n consists in introducing a Jacobi weight on $[0,1]$ in the associated scalar product

$$\langle f, g \rangle = \int_0^1 w_{\alpha,\beta}(t)f(t)g(t)dt, \quad w_{\alpha,\beta}(t) = t^\alpha(1-t)^\beta, \quad for \ \alpha, \beta > -1$$

The extended Durrmeyer-Jacobi operator ([6],[61]) is then defined by

$$M_n^{(\alpha,\beta)}f = \sum_{i=0}^{n} \frac{\langle f, b_i^{(n)} \rangle}{\langle e_0, b_i^{(n)} \cdot \rangle} b_i^{(n)}$$

The limit case $(\alpha, \beta) = (-1, -1)$, corresponding to the weight $\tilde{w}(x) = \dfrac{1}{x(1-x)}$, gives a QI with very attractive properties. It has been introduced by Goodman and Sharma [38][39] for polynomial (and a variant for spline) QIs. It can be written as follows, with $Lf(x) = (1-x)f(0) + xf(1)$:

$$G_n f = Lf + (n-1)\sum_{i=1}^{n-1} \langle f - Lf, b_{i-1}^{(n-2)} \rangle b_i^{(n)}.$$

This operator is exact on $\mathbb{P}_1$ and its behaviour is quite similar to that of the classical Bernstein operator. For example, one has for $f \in C^2(I)$

$$\lim n(f(x) - G_n f(x)) = x(1-x)f''(x).$$

It also preserves the positivity, the monotonicity and the convexity of f. As discrete Bernstein operators, the above operators G_n have associated QIs in the sense of section 3.1 [70] .

2.5 Extrapolation

All operators $\mathcal{B}_n$ described in this section have asymptotic expansions of type

$$\mathcal{B}_n f(x) \sim f(x) + \sum_{k \geq r} \frac{\varphi_k^{(n)}(f, x)}{n^k}$$

for some index r, the $\varphi_k^{(n)}(f, x)$ being linear differential operators depending on n and k. Therefore they are good candidates for extrapolation methods (see e.g. [14] and [82]). Numerical experiments done by the author show that Richardson extrapolation is efficient while the use of variants of epsilon or Δ^2 algorithms often introduce spurious poles in the interval of definition.

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3 Polynomial QIs on a simplex

3.1 Bernstein operator and associated QIs

The simplex S of dimension $d - 1$ is defined in barycentric coordinates as
$S = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) : |\mathbf{x}| = 1\}$ with $|\mathbf{x}| = \sum_{i=1}^{d} |x_i|$.
The associated simplex of indices, monomials and partial derivatives are defined by $\Sigma_n = \{\mathbf{i} = (i_1, i_2, \ldots, i_d) : |\mathbf{i}| = n\}$, $X_n = \{\frac{\mathbf{i}}{n} : \mathbf{i} \in \Sigma_n\} \subset S$,
$\mathbf{i}! = i_1! i_2! \ldots, i_d!$, $\quad \mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d}$, $D^{\mathbf{i}} = D_1^{i_1} D_2^{i_2} \ldots D_d^{i_d}$ with $D_s = \frac{\partial}{\partial x_s}$.
The Bernstein basis of $\mathbb{P}_n$ (space of polynomials of total degree at most n) and the Bernstein operator are defined respectively by:

$$b_{\mathbf{i}}^{(n)}(\mathbf{x}) = \frac{n!}{\mathbf{i}!} \mathbf{x}^{\mathbf{i}} \quad for \ \mathbf{i} \in \Sigma_n, \ \ B_n f(\mathbf{x}) = \sum_{\mathbf{i} \in \Sigma_n} f(\frac{\mathbf{i}}{n}) b_{\mathbf{i}}^{(n)}(\mathbf{x})$$

As $\sum_{\mathbf{i} \in \Sigma_n} b_{\mathbf{i}}^{(n)}(\mathbf{x}) = 1$ and $\sum_{\mathbf{i} \in \Sigma_n} \frac{i_s}{n} b_{\mathbf{i}}^{(n)}(\mathbf{x}) = \mathbf{x}^{\varepsilon_s} = x_s$, for $1 \leq s \leq d$, where
$\varepsilon_s = (0, 0, \ldots, 1, \ldots, 0)$, then B_n is exact on $\mathbb{P}_1$.
Let $l_{\mathbf{i}}^n, \mathbf{i} \in \mathbf{\Sigma_n}\}$ be the Lagrange basis of $\mathbb{P}_n$ associated with the data points X_n.
Then $l_{\mathbf{i}}^n(\frac{\mathbf{j}}{n}) = \delta_{\mathbf{ij}}$ implies $B_n l_{\mathbf{i}}^n = b_{\mathbf{i}}^{(n)}$, hence B_n is an isomorphism of $\mathbb{P}_n$.
For $f \in C^2(S)$, we have the Voronovskaja type result ([47][79][85]),

$$\lim n [B_n f - f] = \frac{1}{2} \bar{\mathcal{D}} f$$

where $\mathcal{D} f$ is the differential operator

$$\bar{\mathcal{D}} f(x) = \sum_{i < j} x_i x_j (\partial_i - \partial_j)^2.$$

B_n and its inverse $A_n = B_n^{-1}$ in $\mathbb{P}_n$ can be expressed as linear differential operators

$$B_n = \sum_{\mathbf{i} \in \mathbf{\Sigma_n}} \beta_{\mathbf{i}}^{(n)} D^{\mathbf{i}}, \quad A_n = \sum_{\mathbf{i} \in \mathbf{\Sigma_n}} \alpha_{\mathbf{i}}^{(n)} D^{\mathbf{i}}$$

whose coefficients can be computed by recursion. For $0 \le k \le n$, define partial inverses

$$A_n^{(k)} = \sum_{\mathbf{i} \in \mathbf{\Sigma_k}} \alpha_{\mathbf{i}}^{(n)} D^{\mathbf{i}}.$$

As in section 2.2 for univariate QIs, we can consider the two families of operators: left Bernstein quasi-interpolants (LBQIs) $B_n^{(k)} = A_n^{(k)} \circ B_n$, and right Bernstein quasi-interpolant (RBQIs) $B_n^{[k]} = B_n \circ A_n^{(k)}$, where $B_n^{(0)} = B_n^{[0]} = B_n =$ and $B_n^{(n)} = B_n^{[n]} = \mathcal{L}_n =$ Lagrange interpolation on X_n.

We have proved [65] that $\|B_n^{(2)}\|_\infty \le 2d+1$ for all $n \ge 2$, and we conjecture that for all $k \ge 0$, there exists a constant $C_k(d)$ such that for all $n \ge k$,

$$\|B_n^{(k)}\|_\infty \le C_k(d).$$

We also conjecture the Voronovskaja-type results

$$\lim n^{r+1}(B_n^{(2r)}f - f) = \mathcal{A}_{2r}f, \quad \lim n^{r+1}(B_n^{(2r+1)}f - f) = \mathcal{A}_{2r+1}^*f,$$

where $\mathcal{A}_{2r}$ and $\mathcal{A}_{2r+1}^*$ are linear differential operators, and the asymptotic expansions

$$B_n^{(2r)}f \text{ and } B_n^{(2r+1)}f \sim f + \frac{c_{r+1}}{n^{r+1}} + \frac{c_{r+2}}{n^{r+2}} \cdots$$

3.2 Durrmeyer-Jacobi QIs on a simplex

One can introduce a Jacobi weight on the simplex in the scalar product of $L_w^2(S)$: $w_\alpha(x) = x^\alpha$, $\langle f, g \rangle = \int_S w_\alpha(x) f(x) g(x) dx$, and define the Durrmeyer-Jacobi quasi-interpolants (DJQIs)

$$M_n f = \sum_{i \in \Sigma_n} \frac{\langle f, b_i^{(n)} \rangle}{\langle e_0, b_i^{(n)} \rangle} b_i^{(n)}.$$

Its eigenvectors are the Jacobi polynomials on the simplex. There holds a Voronovskaja type result [13][79]

$$\lim n(M_n f(x) - f(x)) = \mathcal{D}_\alpha f(x)$$

where the differential operator $\mathcal{D}_\alpha$ is defined by

$$\mathcal{D}_\alpha = x^{-\alpha} \sum_{i<j} (\partial_i - \partial_j) x_i x_j^\alpha (\partial_i - \partial_j).$$

As M_n is an isomorphism of $\mathbb{P}_n$, one can expand $M_n = \sum_{k=0}^n \sum_{\mathbf{i} \in \mathbf{\Sigma_k}} \beta_{\mathbf{i}}^{(n)} D^{\mathbf{i}}$ and $L_n = M_n^{-1} = \sum_{k=0}^n \sum_{\mathbf{i} \in \mathbf{\Sigma_n}} \alpha_{\mathbf{i}}^{(n)} D^{\mathbf{i}}$. As in the univariate case [71], the polynomials $\beta_{\mathbf{i}}^{(n)}$ and $\alpha_{\mathbf{i}}^{(n)}$ are probably linear combinations of Jacobi polynomials on S ([26]) Setting $L_n^{(r)} = \sum_{k=0}^r \sum_{\mathbf{i} \in \mathbf{\Sigma_n}} \alpha_{\mathbf{i}}^{(n)} D^{\mathbf{i}}$, one can define the left DJQIs $M_n^{(r)} = L_n^{(r)} \circ M_n$, and the right DJQIs $M_n^{[r]} = M_n \circ L_n^{(r)}$, with $M_n^{(0)} = M_n$ and $M_n^{(n)} = P_n = $ orthogonal projector on $\mathbb{P}_n$ in $L^2(S)$. They have the same properties as univariate QIs, and it would be interesting to have detailed proofs, those of [65][66] being only sketched. However, the author thinks that the following operators are still more attractive.

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3.3   Jetter-Stöckler operators on a triangle

For the sake of simplicity, we describe them over a triangle (with barycentric coordinates $\{\lambda_1, \lambda_2, \lambda_3\}$) in the case of the Legendre weight ($w = 1$, see [42] for the general study on a simplex with Jacobi weight). Using the following notations:

$$D_{ij} = \partial_j - \partial_i, \ i < j, \ \ D = \{D_{12}, D_{13}, D_{23}\}, \ \Lambda = \{\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3\},$$

$$\mathbf{k} = (k_{12}, k_{13}, k_{23}) \in \mathbb{N}^3, \ \mathcal{D}^{\mathbf{k}} = D_{12}^{k_{12}} D_{13}^{k_{13}} D_{23}^{k_{23}},$$

$$\Lambda^{\mathbf{k}} = (\lambda_1\lambda_2)^{k_{12}}(\lambda_1\lambda_3)^{k_{13}}(\lambda_2\lambda_3)^{k_{23}},$$

the authors define the following basic differential operators:

$$U_{\mathbf{k}} = \frac{1}{\mathbf{k}!}(-1)^{|\mathbf{k}|} \mathcal{D}^{\mathbf{k}} \Lambda^{\mathbf{k}} \mathcal{D}^{\mathbf{k}}, \ \ \mathcal{U}_\ell = \frac{1}{\ell!} \sum_{|\mathbf{k}|=\ell} U_{\mathbf{k}}, \ \ \mathcal{Y}_n = \sum_{\ell=0}^n (C_n^\ell)^{-1} \mathcal{U}_\ell$$

Let M_n be the Durrmeyer operator, then they prove that $U_{\mathbf{k}}$ commute with M_n for all pairs $(\mathbf{k}, n)$ and that $\mathcal{Y}_n$ is the inverse of M_n in the space of polynomials $\mathbb{P}_n$. Now, for $0 \le r \le n$ fixed, they define partial inverses and left Jetter-Stöckler quasi-interpolants (LJSQIs)

$$\mathcal{Y}_n^{(r)} = \sum_{\ell=0}^r (C_n^\ell)^{-1} \mathcal{U}_\ell, \ \ M_n^{(r)} = \mathcal{Y}_n^{(r)} M_n.$$

One can also define right JSQIs $M_n^{[r]} = M_n \mathcal{Y}_n^{(r)}$. Both operators $M_n^{(r)}$ and $M_n^{[r]}$ are exact on $\mathbb{P}_r$. Moreover, for r fixed, the left JSQIs have uniformly bounded infinite norms w.r.t. n. Finally, the authors prove Voronovskaja-type results:

$$\lim_{n \to \infty} C_n^r(f - M_n^{(r-1)}f) = \mathcal{U}_r f$$

3.4 Extrapolation

All operators $\mathcal{B}_n$ described in this section have asymptotic expansions of type

$$\mathcal{B}_n f(x) \approx f(x) + \sum_{k \geq r} \frac{\varphi_k^{(n)}(f, x)}{n^k}$$

(see e.g. [47] and [85]). In particular, the latter reports interesting numerical results on Richardson extrapolation of classical Bernstein operators on the triangle. It would be interesting to compare these results with those which could be obtained by extrapolating the above QIs.
PARTITIONS————————————

4 Univariate spline QIs on uniform partitions

4.1 Univariate differential and discrete QIs

For the construction of QIs with optimal approximation order, we refer to [15] and [16], where general solutions are given, thus completing the initial work by Schoenberg in [80].

4.2 Near-best spline dQIs

Consider the family of spline dQIs of order $2m$ depending on $n + 1$ arbitrary parameters $a = (a_0, a_1, \ldots, a_n)$, $n \geq m$:

$$Q_a f = \sum_{i \in \mathbb{Z}} \Lambda f(i) M_{2m}(x - i)$$

with coefficient functionals

$$\Lambda f(i) = a_0 f(i) + \sum_{j=1}^{n} a_j \left(f(i+j) + f(i-j) \right).$$

Setting $\nu(a) = |a_0| + \sum_{j=1}^{n} |a_j|$, then we have $\|Q_a\|_\infty \leq \nu(a)$. By imposing that Q_a be exact on $\mathbb{P}_r$, with $0 \leq r \leq 2m - 1$, we obtain a set of linear constraints: $a \in V_r \subset \mathbb{R}^{n+1}$. We say that $Q^* = Q_{a^*}$ is a *near best dQI* if

$$\nu(a^*) = \min\{\nu(a); a \in V_r\}.$$

There is existence, but in general not unicity, of solutions.
Example: cubic splines (see [40]). There is a unique optimal solution for $n \geq 2$:

$$a_0^* = 1 + \frac{1}{3n^2}, \ \ a_n^* = -\frac{1}{6n^2}, \ \ a_j^* = 0 \ \ for \ \ 1 \leq j \leq n - 1$$

Moreover, for all $n \geq 4$, $\|Q^*\|_\infty \leq 1 + \frac{2}{3n^2}$. Here are the first values of $\|Q^*\|_\infty$ and $\nu(a^*)$; $n = 1 : 1.222$ & 1.666; $n = 2 : 1.139$ & 1.166; $n = 3 : 1.074$ & 1.074.
PARTITIONS————————————

4.3   Near-best spline iQIs

A similar study can be done for integral spline QIs. We refer to [2][40] and we only give an example given in these papers. Setting $a = (a_0, a_1, \ldots, a_n)$, $n \geq m$ and $M_i(x) = M_{2m}(x-i)$, we consider $Q_a f = \sum_{i \in \mathbb{Z}} \Lambda f(i) M_i$ with coefficient functionals

$$\Lambda f(i) = a_0 \langle f, M_i \rangle + \sum_{j=1}^{n} a_j \left(\langle f, M_{i-j} \rangle + \langle f, M_{i+j} \rangle \right).$$

As in section 4.2, we have $\|Q_a\|_\infty \leq \nu(a)$ and we say that $Q^* = Q_{a^*}$ is a *near best iQI* if $\nu(a^*) = \min\{\nu(a); a \in V_r\}$. There is existence, but in general not unicity, of solutions.

Example: cubic splines (see [40]). There is a unique optimal solution for $n \geq 2$:

$$a_0^* = 1 + \frac{2}{3n^2}, \quad a_n^* = -\frac{1}{3n^2}, \quad a_j^* = 0 \ \ for \ \ 1 \leq j \leq n-1$$

Moreover, for all $n \geq 4$, $\|Q^*\|_\infty \leq 1 + \frac{4}{3n^2}$. Here are the first values of $\|Q^*\|_\infty$ and $\nu(a^*)$; $n = 1 : 1.5278$ & 2.333; $n = 2 : 1.2778$ & 1.333; $n = 3 : 1.1481$ & 1.1482.

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5   Bivariate spline dQIs on uniform partitions

5.1   A general construction of dQIs

Let φ be any kind of bivariate B-spline on one of the two classical three- or four-directional meshes of the plane (e.g. box-splines, see [7],[12],[19]). Let $\Sigma = supp(\varphi)$ and $\Sigma^* = \Sigma \cap \mathbb{Z}^2$. Let a be the hexagonal (or lozenge=rhombus) sequence formed by the values $\{\varphi(i), i \in \Sigma^*\}$. The associated central difference operator $\mathcal{D}$ is an isomorphism of $\mathbb{P}(\varphi)$, the maximal subspace of "complete " polynomials in the space of splines $\mathcal{S}(\varphi)$ generated by the integer translates of the B-spline φ (see [12], [69],[71],[72]). Computing the expansion of a in some basis of the space of hexagonal (or lozenge) sequences amounts to expand $\mathcal{D}$ in some basis of central difference operators. Then, computing the formal inverse $\mathcal{D}^{-1}$ allows to define the dQI

$$\mathcal{Q}f = \sum_{k \in \mathbb{Z}^2} \mathcal{D}^{-1} f(k) \varphi(\cdot - k)$$

which is exact on $\mathbb{P}(\varphi)$. Let us now give two examples which are detailed in [40].

5.2   Near-best spline dQIs on a three direction mesh

Example: let φ be the C^2 quartic box-spline. Let H_s be the regular hexagon with edges of length $s \geq 1$, centered at the origin (here $\Sigma = H_2$) and let $H_s^* = H_s \cap \mathbb{Z}^2$. The near-best dQIs have coefficient functionals with supports consisting of the

center and the 6 vertices of $H_s^*, s \geq 1$. The coefficients of values of f at those points are respectively $1 + \frac{1}{2s^2}$ and $-\frac{1}{12s^2}$, therefore the infinite norm of the optimal dQIs Q_s^* is bounded above by $\nu_s^* = 1 + \frac{1}{s^2}$. Here are the first values of $\|Q^*\|_\infty$ and ν_s^*; $n = 1 : 1.34028 \ \& \ 2; n = 2 : 1.22917 \ \& \ 1.25; n = 3 : 1.10185 \ \& \ 1.111$.

5.3 Near-best spline dQIs on a four direction mesh

Example: let φ be the C^1 quadratic box-spline. Let Λ_s be the lozenge (rhombus) with edges of length $s \geq 1$, centered at the origin, and let $\Lambda_s^* = \Lambda_s \cap \mathbb{Z}^2$. The near-best dQIs have coefficient functionals with supports consisting of the center and the 4 vertices of $\Lambda_s^*, s \geq 1$. The coefficients of values of f at those points are respectively $1 + \frac{1}{2s^2}$ and $-\frac{1}{8s^2}$, therefore the infinite norm of the optimal dQIs Q_s^* is bounded above by $\nu_s^* = 1 + \frac{1}{s^2}$. Here are the first values of $\|Q^*\|_\infty$ and ν_s^*; $n = 1 : 1.5 \ \& \ 2; n = 2 : 1.25 \ \& \ 1.25; n = 3 : 1.111 \ \& \ 1.111$.

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6 Univariate spline QIs on non uniform partitions

6.1 Uniformly bounded dQIs

Let us only give an example: we start from a family of DQIs of degree m which are *exact on* $\mathbb{P}_2$.

$$Q_2 f = \sum_{j \in J} \lambda_j^{(2)}(f) B_j, \ \ \lambda_j^{(2)}(f) = f(\theta_j) - \frac{1}{2}(\theta_j^2 - \theta_j^{(2)}) D^2 f(\theta_j).$$

We recall the expansion [52][53]

$$A_j^{(2)} = \theta_j^2 - \theta_j^{(2)} = \frac{1}{(m-1)^2(m-2)} \sum_{(r,s) \in E_m^2, r \neq s} (t_{j+r} - t_{j+s})^2 > 0.$$

On the other hand, $\frac{1}{2} D^2 f(\theta_j)$ can be replaced on the space $\mathbb{P}_2$ by the second order divided difference $[\theta_{j-1}, \theta_j, \theta_{j+1}]f$, therefore the dQI defined by

$$Q_2^* f = \sum_{j \in J} \mu_j^{(2)}(f) B_j, \ \ \mu_j^{(2)}(f) = f(\theta_j) - A_j^{(2)}[\theta_{j-1}, \theta_j, \theta_{j+1}]f,$$

is also exact on $\mathbb{P}_2$. Moreover, one can write

$$\mu_i^{(2)}(f) = a_i f_{i-1} + b_i f_i + c_i f_{i+1}$$

with $a_i = -A_i^{(2)}/\Delta\theta_{i-1}(\Delta\theta_{i-1} + \Delta\theta_i)$, $c_i = -A_i^{(2)}/\Delta\theta_i(\Delta\theta_{i-1} + \Delta\theta_i)$, and $b_i = 1 + A_i^{(2)}/\Delta\theta_{i-1}\Delta\theta_i$, So, according to the introduction

$$\|Q_2^*\|_\infty \leq \max_{i \in J}(|a_i| + |b_i| + |c_i|) \leq 1 + 2 \max_{i \in J} \frac{A_i^{(2)}}{\Delta\theta_{i-1}\Delta\theta_i}.$$

The following theorem [4] extends a result given for quadratic splines in [4][73][75].

Theorem 1. *For any degree m, the dQIs Q_2^* are UB. More specifically, for all partitions of I:*

$$\|Q_2^*\|_\infty \le [\frac{1}{2}(m+4)]$$

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6.2 Uniformly bounded iQIs

General types of integral QIs are studied in [21][63][68]. Here, we have chosen to study a family of QIs that we call Goodman-Sharma type iQIs, as they first appear in [38]. They seem simpler and more interesting than those we have studied in [68]. The simpler GS-type IQI can be written as follows

$$G_1 f = f(t_0)B_0 + \sum_{i=1}^{n+m-2} \tilde{\mu}_i(f)B_i + f(t_n)B_{n+m-1},$$

where the integral coefficient functionals are defined by

$$\tilde{\mu}_i(f) = \int_0^1 \tilde{M}_{i-1}(t)f(t)dt,$$

$\tilde{M}_{i-1}(t)$ being the B-spline of degree $m-2$ with support $\tilde{\Sigma}_{i-1} = [t_{i-m+1}, t_i]$, normalized by $\tilde{\mu}_i^{(0)} = \tilde{\mu}_i(e_0) = \int_0^1 \tilde{M}_{i-1}(t) = 1$. It is easy to verify that G_1 is exact on $\mathbb{P}_1$ and that $\|G_1\|_\infty = 1$. We shall study the family of GS-type iQIs defined by

$$G_2 f = f(t_0)B_0 + \sum_{i=1}^{n+m-2} [a_i\tilde{\mu}_{i-1}(f) + b_i\tilde{\mu}_i(f) + c_i\tilde{\mu}_{i+1}(f)]B_i + f(t_n)B_{n+m-1},$$

which are *exact* on $\mathbb{P}_2$. The three constraints $G_2 e_k = e_k, \quad k = 0, 1, 2$, lead to the following system of equations, for $1 \le i \le n + m - 2$:

$$a_i + b_i + c_i = 1, \quad \theta_{i-1}a_i + \theta_i b_i + \theta_{i+1}c_i = \theta_i, \quad \tilde{\mu}_{i-1}^{(2)}a_i + \tilde{\mu}_i^{(2)}b_i + \tilde{\mu}_{i+1}^{(2)}c_i = \theta_i^{(2)}.$$

This is a consequence of the following facts

$$\tilde{\mu}_i(e_1) = \int_0^1 t\tilde{M}_{i-1}(t)dt = \frac{1}{m}\sum_{s=1}^m t_{i-m+s} = \theta_i,$$

$$\tilde{\mu}_i^{(2)} = \mu_i(e_2) = \int_0^1 t^2\tilde{M}_{i-1}(t)dt = \frac{2}{m(m+1)}\tilde{s}_2(T_i)$$

$$= \frac{2}{m(m+1)} \sum_{1 \le r \le s \le m} t_{i-m+r}t_{i-m+s}$$

Theorem 2. *For any degree m, the iQIs G_2 are UB. More specifically, for all partitions of I:*

$$\|G_2\|_\infty \le 5$$

The detailed proof will be given in [78].

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6.3   Near-best dQIs

Let us consider the family of dQIs of degree m defined, for the sake of simplicity, on $I = \mathbb{R}$ endowed with an arbitrary non-uniform increasing sequence of knots $T = \{t_i; i \in \mathbb{Z}\}$, by

$$Qf = Q_{p,q}f = \sum_{i\in\mathbb{Z}} \mu_i(f)B_i.$$

Their coefficient functionals depend on $2p + 1$ parameters, with $p \ge m$:

$$\mu_i(f) = \sum_{s=-p}^{p} \lambda_i(s)f(\theta_{i+s}),$$

and they are exact on the space $\mathbb{P}_q$, where $q \le \min(m, 2p)$. The latter condition is equivalent to $Qe_r = e_r$ for all monomials of degrees $0 \le r \le q$. It implies that for all indices i, the parameters $\lambda_i(s)$ satisfy the system of $q + 1$ linear equations:

$$\sum_{s=-p}^{p} \lambda_i(s)\theta_{i+s}^r = \theta_i^{(r)}, \quad 0 \le r \le q.$$

The matrix $V_i \in \mathbb{R}^{(q+1)\times(2p+1)}$ of this system, with coefficients $V_i(r,s) = \theta_{i+s}^r$, is a Vandermonde matrix of maximal rank $q + 1$, therefore there are $2p - q$ *free parameters*. Denoting $b_i \in \mathbb{R}^{q+1}$ the vector in the right hand side, with components $b_i(r) = \theta_i^{(r)}, \quad 0 \le r \le q$, we consider the sequence of minimization problems, for $i \in \mathbb{Z}$:

$$\min \|\lambda_i\|_1, \quad V_i\lambda_i = b_i.$$

We have seen in the introduction that $\nu_1^*(Q) = \max_{i\in\mathbb{Z}} \min \|\lambda_i\|_1$ is an upper bound of $\|Q_q\|_\infty$ which is easier to evaluate than the true norm of the dQI.

Theorem 3. *The above minimization problems have always solutions, which, in general, are non unique.*

The *objective function being convex* and *the domains being affine subspaces*, these classical optimization problems have always solutions, in general non unique. Example of optimal dQIs are given in [1][4][40].

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7 Bivariate quadratic spline dQIs on non uniform criss-cross triangulations

At the author's knowledge, the only bivariate box-splines which have been extended to non uniform partitions of the plane are C^1-quadratic box-splines on criss-cross triangulations [20][62]. Recently, we have constructed a set of B-splines generating the space of quadratic splines on a rectangular domain and we have defined a discrete quasi-interpolant which is exact on $\mathbb{P}_2$ and uniformly bounded independently of the partition [74]-[76].

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8 Abbreviations for publishers and journals

Publishers: AP=Academic Press, New-York ; BAS=Bulgarian Academy of Science, Sofia; BV=Birkhäuser-Verlag, Basel; CUP=Cambridge University Press; JWS=John Wiley & Sons, New-York; K=Kluwer, Dordrecht; NH= North-Holland, Amsterdam; NP=Nashboro Press, Brentwood; SV=Springer-Verlag, Berlin; SIAM=Society for Industrial and Applied Mathematics, Philadelphia; VUP=Vanderbilt University Press, Nashville.
Journals: AiCM=Advances in Comput. Mathematics; ATA=Approximation Theory and its Applications (now Analysis in Theory and Applications); CAGD=Computer Aided Geometric Design; JAT=Journal of Approximation Theory; JCAM=Journal of Computational and Applied Mathematics.
Proceedings: AT2=*Approximation Theory II*, G.G. Lorentz, C.K. Chui, L.L. Schumaker (eds), AP 1976; AT4 & AT5=Approximation Theory IV & V, C.K. Chui, L.L. Schumaker, J.D. Ward (eds), AP 1983 and 1986; CMSB=Colloquia Mathematica Soc. Janos Bolyai; CS02=Curve and Surface Fitting (St Malo 2002), A. Cohen, J.L. Merrien and L.L. Schumaker (eds), NP 2003.
Preprints: PI=Prépublications IRMAR, Inst. de Recherche Math. de Rennes.

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